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CONSTRUCTIVE APPROXIMATION

Sampling Almost Periodic and Related Functions

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Abstract
We consider certain finite sets of circle-valued functions defined on intervals of real numbers and estimate how large the intervals must be for the values of these functions to be uniformly distributed in an approximate way. This is used to establish some general conditions under which a random construction introduced by Katznelson for the integers yields sets that are dense in the Bohr group. We obtain in this way very sparse sets of real numbers (and of integers) on which two different almost periodic functions cannot agree, which makes them amenable to be used in sampling theorems for these functions. These sets can be made as sparse as to have zero asymptotic density or as to be t-sets, i.e., to be sets that intersect any of their translates in a bounded set. Many of these results are proved not only for almost periodic functions but also for classes of functions generated by more general complex exponential functions, including chirps or polynomial phase functions.

Keywords Almost periodic function · Bohr topology · Matching set · Sampling set · Chirp · Polynomial phase functions · Discrepancy · Uniform distribution · Sampling · T-set · Bohr-dense

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1 Introduction

Many sampling processes depend on choosing a sampling set on which the functions to be sampled are uniquely determined. In the case of almost periodic functions on

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$\mathbb{R}$, sets with that property admit a neat topological description: they are precisely those subsets of $\mathbb{R}$ that are dense in the Bohr topology. A clear separation between consecutive samples is another natural prerequisite, and, for this reason, sampling sets are usually required to be uniformly discrete. Although Bohr-density and uniform discreteness might seem conflicting requirements, they are not totally incompatible: Bohr-dense sets can be not only uniformly discrete but even quite sparse, as we next describe.

Collet [7] (see also Carlen and Mendes [6] for a similar approach to functions that can be approximated by polynomial phase functions) proved that a random selection of points in regularly spaced time-windows almost surely produces a set that is uniformly distributed in the Bohr compactification, hence is dense in the Bohr topology (see the section below for unexplained terms), and whose asymptotic density is as small as desired.

Motivated by a different sort of problem, Katznelson [13] devised a method that almost surely produces subsets of the group of integers $\mathbb{Z}$ with asymptotic null density which are dense in the Bohr topology. These constructions were further developed in [11, 14].

In this paper we lay out general conditions for Katznelson's method to work both in $\mathbb{R}$ and $\mathbb{Z}$ for almost periodic functions and, more generally, for function spaces generated by complex exponentials $e^{2\pi i p(t)}$ with $p(t)$ running over sets of polynomials of bounded degree with coefficients in $\mathbb{R}$ (usually known as *chirps* or *polynomial phase functions*). Namely, we prove that partitioning $\mathbb{R}$ into intervals of increasing length and then choosing ℓ_k random points in each partition, with ℓ_k larger than the order of the logarithm of the size of the corresponding cell of the partition, we almost surely obtain a set where these functions are uniquely determined, a set of uniqueness for them, see Definition 2.1.

The dense sets we obtain, as those in [6, 7, 13], are obtained through the Borel–Cantelli lemma and might require extremely large intervals to be reliable. Our approach gives hints on the minimum size a sampling interval must have in order to use our methods for approximate sampling.

Once our general construction is laid, dense subsets of the Bohr group with specific properties are easy to obtain. In order to illustrate this, we show the existence of t-sets that are dense in the Bohr compactification. By a t-set we refer here to a special sort of *thin* sets, introduced by Rudin [17], which are sets of interpolation for the weakly almost periodic functions, see Sect. 5 for more on this.

1.1 Notation and Terminology

Even if most of our results are proved for $G = \mathbb{R}$ or $G = \mathbb{Z}$, we find it more convenient to state the results of Sect. 2 for general topological Abelian groups or for locally compact Abelian (LCA) groups, depending on whether they involve probabilities or not. In this latter case probabilities will be computed using the Haar measure λ_G of G. When it comes to $G = \mathbb{R}$ and $G = \mathbb{Z}$, we assume that Haar measures are normalized so that λ_G is the Lebesgue measure. If $I \subseteq \mathbb{R}$ is an interval, $\lambda_{\mathbb{R}}(I)$ therefore corresponds to the length of I.

In many of our proofs, we consider a random choice Λ of, say, ℓ-many elements of a subset I of a given locally compact group G, and we estimate the probability that Λ belongs to a measurable subset $\mathcal{A}$ of I^ℓ that is invariant under permuting coordinates. This random choice, and its corresponding probability, is always to be understood in the probability space induced by λ_{G^ℓ} on I^ℓ. Hence for a given $\mathcal{A} \subseteq I^\ell$, we have that $\mathbb{P}(\{\Lambda \colon \Lambda \in \mathcal{A}\}) = \lambda_{G^\ell}(\mathcal{A})/\lambda_G(I)^\ell$.

When it comes to duality, the circle group $\mathbb{T}$ has a central rôle. When a specific distance on $\mathbb{T}$ is required, our choice is the *angular distance* defined for every $t,\ s \in [0, 1)$ by

$$d_a\left(e^{2\pi i t}, e^{2\pi i s}\right) = \min\{|t - s|, 1 - |t - s|\}.$$

The open ball of radius ε and center 1 will be denoted by $\mathcal{V}_\varepsilon$, thus:

$$\mathcal{V}_\varepsilon = \left\{e^{2\pi i t} \in \mathbb{T} \colon |t| < \varepsilon\right\} \subseteq \mathbb{T}.$$

If G is a topological group, $C(G, \mathbb{T})$ denotes the multiplicative group of continuous $\mathbb{T}$-valued functions, and $\widehat{G}$ its subgroup of continuous homomorphisms. We refer to $\widehat{G}$ as the *character group* of G. Characters of $\mathbb{R}$ are denoted by $\chi_\tau,\ \tau \in \mathbb{R}$ where $\chi_\tau(s) = e^{2\pi i \tau s}$, for every $s \in \mathbb{R}$. It is a standard fact that the map $\tau \longmapsto \chi_\tau$ establishes an isomorphism that maps $\mathbb{R}$ onto $\widehat{\mathbb{R}}$. The same mapping with $\tau \in \mathbb{Z}$ establishes an isomorphism which maps $\mathbb{Z}$ onto $\widehat{\mathbb{T}}$.

If $p(x)$ is a polynomial with real coefficients, we denote by ψ_p the function $\psi_p(t) = e^{2\pi i p(t)}$, so that $\psi_p = \chi_\tau$, when $p(t) = \tau t$. The symbol $\mathfrak{C}_n$ stands for the set $\left\{e^{2\pi i p(t)} \colon p \in \mathbb{R}_n[x]\right\}$, where $\mathbb{R}_n[x]$ denotes the set of polynomials in $\mathbb{R}[x]$ of degree at most n. Functions in $\mathfrak{C}_n$ are known as *chirps* or *polynomial phase functions*.

Almost periodic functions were originally introduced by H. Bohr in the seminal papers [2, 3], and [4] in terms of relatively dense ε-periods; however, for our purposes it is convenient to regard almost periodic functions as functions that can be approximated by linear combinations of characters (known as trigonometric polynomials). In the next definition we extend this approach to spaces generated by other $\mathbb{T}$-valued functions; see [5] or [12] for other definitions, including H. Bohr's original one.

Definition 1.1 Let G be a topological group. For $\mathfrak{J} \subseteq C(G, \mathbb{T})$, let $\operatorname{span}(\mathfrak{J})$ denote the linear span of $\mathfrak{J}$ in the vector space $C(G, \mathbb{C})$. We define

$$\mathcal{AP}_{\mathfrak{J}}(G) = \overline{\operatorname{span}(\mathfrak{J})}^{\|\cdot\|_\infty}.$$

When $\mathfrak{J} = \widehat{G}$ we obtain the space of *almost periodic* functions on G, and we simply denote it by $\mathcal{AP}(G)$.

The topology that almost periodic functions induce on a group is known as the *Bohr topology*. This is the topology the group inherits from its embedding in $\prod_{\chi \in \widehat{G}} \mathbb{T}_\chi$, with $\mathbb{T}_\chi = \mathbb{T}$ for every χ, given by $g \longmapsto (\chi(g))_\chi$ for every $g \in G$. The closure $G^{\mathcal{AP}}$ of G in $\prod_{\chi \in \widehat{G}} \mathbb{T}_\chi$ is known as the Bohr compactification of G and can also be identified with the spectrum of $\mathcal{AP}(G)$ when $\mathcal{AP}(G)$ is viewed as a Banach algebra.

A bounded and continuous function f on G is almost periodic precisely when it can be continuously extended to a function $f^b \in C(G^{\mathcal{AP}})$. Therefore if D is a discrete subset of G that is dense in $G^{\mathcal{AP}}$, then there is at most one almost periodic function f on G that fits any values that were preassigned on D.

We next define the sort of almost periodic functions that are most suitable for our sampling methods, the functions with summable Bohr–Fourier series.

Definition 1.2 Let G be a topological group and $\mathfrak{J} \subseteq C(G, \mathbb{T})$. We define

$$A_{\mathfrak{J}}(G) = \left\{ \sum_{\phi \in \mathfrak{J}} \alpha_\phi \phi \colon \sum_{\phi \in \mathfrak{J}} \left|\alpha_\phi\right| < \infty \right\}.$$

The natural isomorphism between the Banach algebra $\ell_1(\mathfrak{J})$ and $A_{\mathfrak{J}}(G)$ defines a norm on this latter space. We denote this norm by $\|\cdot\|_{A_{\mathfrak{J}}}$.

When $\mathfrak{J} = \widehat{G}$, $A_{\mathfrak{J}}(G)$ can be identified with the Fourier algebra $A(G^{\mathcal{AP}})$ (as defined, for instance, in [18, Section 1.2.3]) on the Bohr compactificaction $G^{\mathcal{AP}}$ of G. Note that $A_{\mathfrak{J}}(G)$ may be strictly contained in $\mathcal{AP}_{\mathfrak{J}}(G)$, see [18, Theorem 4.6.8]; the result is originally due to Segal [19].

If X is a set, a subset $A \subseteq X$ is an *n-subset* if its cardinality is $n \in \mathbb{N}$. A subset D of a metric space (X, ρ) is *ε-dense* if for every $x \in X$ there exists $d_x \in D$ such that $\rho(x, d_x) < \varepsilon$.

2 Approximately Bohr-Dense Subsets

We aim to construct sets where particular families of functions are uniquely determined.

Definition 2.1 Let Λ be a subset of a topological group G, and let $\mathcal{A}$ be a vector subspace of $C(G, \mathbb{C})$. We say that Λ is a *set of uniqueness* for $\mathcal{A}$ if whenever $f \in \mathcal{A}$ satisfies $f\big|_\Lambda = 0$, then $f = 0$.

We first consider sets of approximate uniqueness.

Definition 2.2 Let G be a topological group. Fix $I \subseteq G$ and $\mathfrak{J} \subseteq C(G, \mathbb{T})$. The subset $\Lambda \subseteq I$ is an *$(\mathfrak{J}, I, \varepsilon)$-sampling set* if for every $f \in A_{\mathfrak{J}}(G)$,

$$\left\| f\big|_I \right\|_\infty \leq \varepsilon \cdot \|f\|_{A_{\mathfrak{J}}} + \left\| f\big|_\Lambda \right\|_\infty.$$

The following sets contain enough elements to approximate the values of a given family F of functions in $C(G, \mathbb{T})$.

Definition 2.3 Let G be a topological group. Fix $F \subseteq C(G, \mathbb{T})$, $I \subseteq G$ and $\varepsilon > 0$. The subset $\Lambda \subseteq G$ is an *(F, I, ε)-matching set* if for every $a \in I$ there exists $x_a \in \Lambda$ such that $\phi(x_a) \in \phi(a) \cdot \mathcal{V}_\varepsilon$ for every $\phi \in F$.

The notation below provides a compact expression of the property defining matching sets.

Definition 2.4 For $I \subseteq G$, $a \in G$, $\Delta \subseteq C(G, \mathbb{T})$ and $\varepsilon > 0$, we define

(1) $N^{\triangleright}(I, \varepsilon) = \{\phi \in C(G, \mathbb{T}) \colon \phi(I) \subseteq \mathcal{V}_\varepsilon\}$, and
(2) $N^{\triangleleft}(\Delta, \varepsilon, a) = \{t \in G \colon \phi(t) \in \phi(a) \cdot \mathcal{V}_\varepsilon$ for all $\phi \in \Delta\}$. For $a = 0$ we write $N^{\triangleleft}(\Delta, \varepsilon)$.

Observe that $N^{\triangleleft}(\Delta, \varepsilon, a) = a + N^{\triangleleft}(\Delta, \varepsilon)$ in case Δ consists of homomorphisms. In these terms, the subset $\Lambda \subseteq G$ is an (F, I, ε)-*matching set* if $\Lambda \cap N^{\triangleleft}(F, \varepsilon, a) \neq \varnothing$ for every $a \in I$. We now see that matching sets, even when slightly thickened, are sampling sets.

Proposition 2.5 *Let G be a topological group, $\Lambda \subseteq I \subseteq G$ and $\mathfrak{J} \subseteq C(G, \mathbb{T})$. Suppose that there exists $F \subseteq C(G, \mathbb{T})$ that satisfies*

(1) *Λ is an (F, I, ε)-matching set, and*
(2) *$\mathfrak{J} \subseteq F \cdot N^{\triangleright}(I, \varepsilon)$.*

Then Λ is a $(\mathfrak{J}, I, (6\pi + 2)\varepsilon)$-sampling set.

Proof For $f \in A_{\mathfrak{J}}(G)$, let $(\phi_j)_{j=1}^n \subseteq \mathfrak{J}$ and $(\alpha_j)_{j=1}^n \subseteq \mathbb{C}$ be such that $\left\| f - \sum_{j=1}^n \alpha_j \phi_j \right\|_\infty < \varepsilon \cdot \|f\|_{A_{\mathfrak{J}}}$. Let $z \in I$ be fixed. Given that $\mathfrak{J} \subseteq F \cdot N^{\triangleright}(I, \varepsilon)$, we can choose $\psi_j \in F$ and $\kappa_j \in N^{\triangleright}(I, \varepsilon)$ with $\phi_j = \psi_j \cdot \kappa_j$. Since $\kappa_j \in N^{\triangleright}(I, \varepsilon)$ implies $|\kappa_j(z) - 1| < 2\pi\varepsilon$, we obtain

$$\left| \sum_{j=1}^n \alpha_j \phi_j(z) - \sum_{j=1}^n \alpha_j \psi_j(z) \right| \leq \sum_{j=1}^n |\alpha_j| \cdot \left|\psi_j(z)\right| \cdot \left|\kappa_j(z) - 1\right| < 2\pi\varepsilon \cdot \|f\|_{A_{\mathfrak{J}}}.$$

On the other hand, since Λ is an (F, I, ε)-matching set, there exists $x_z \in \Lambda$ such that $\psi(x_z) \in \psi(z) \cdot \mathcal{V}_\varepsilon$ for every $\psi \in F$. Hence

$$\left| \sum_{j=1}^n \alpha_j \psi_j(z) - \sum_{j=1}^n \alpha_j \psi_j(x_z) \right| \leq \sum_{j=1}^n |\alpha_j| \cdot \left|\psi_j(z) - \psi_j(x_z)\right| \leq 2\pi\varepsilon \cdot \|f\|_{A_{\mathfrak{J}}}.$$

Therefore, from

$$\begin{aligned} |f(z)| \leq & \left| f(z) - \sum_{j=1}^n \alpha_j \phi_j(z) \right| + \left| \sum_{j=1}^n \alpha_j \phi_j(z) - \sum_{j=1}^n \alpha_j \psi_j(z) \right| \\ & + \left| \sum_{j=1}^n \alpha_j \psi_j(z) - \sum_{j=1}^n \alpha_j \psi_j(x_z) \right| + \left| \sum_{j=1}^n \alpha_j \psi_j(x_z) - \sum_{j=1}^n \alpha_j \phi_j(x_z) \right| \\ & + \left| \sum_{j=1}^n \alpha_j \phi_j(x_z) - f(x_z) \right| + |f(x_z)|, \end{aligned}$$

we obtain

$$|f(z)| < \varepsilon \cdot \|f\|_{A_{\mathfrak{J}}} + 2\pi\varepsilon \cdot \|f\|_{A_{\mathfrak{J}}} + 2\pi\varepsilon \cdot \|f\|_{A_{\mathfrak{J}}} + 2\pi\varepsilon \cdot \|f\|_{A_{\mathfrak{J}}} + \varepsilon \cdot \|f\|_{A_{\mathfrak{J}}} + \left\| f\big|_{\Lambda} \right\|_{\infty} = (6\pi + 2)\varepsilon \cdot \|f\|_{A_{\mathfrak{J}}} + \left\| f\big|_{\Lambda} \right\|_{\infty}.$$

□

As may be expected, Proposition 2.5 yields density results when applied to all finite subsets of a concrete subspace of $C(G, \mathbb{T})$.

Corollary 2.6 *Let G be a topological group, $\mathfrak{J} \subseteq C(G, \mathbb{T})$ and $\Lambda \subseteq G$. Suppose that Λ is an (F, G, ε)-matching set for every finite subset $F \subseteq \mathfrak{J}$ and every $\varepsilon > 0$. Then Λ is a set of uniqueness for $\mathcal{AP}_{\mathfrak{J}}(G)$.*

Proof Let $f \in \mathcal{AP}_{\mathfrak{J}}(G)$ with $f\big|_{\Lambda} = 0$, and let $0 < \varepsilon < 1$ be fixed. Find a $\mathfrak{J}$-trigonometric polynomial $P_f = \sum_{j=1}^{n} \alpha_j \psi_j$, with $(\alpha_j)_{j=1}^{n} \subseteq \mathbb{C}$ and $(\psi_j)_{j=1}^{n} \subseteq \mathfrak{J}$, such that $\left\| f - P_f \right\|_{\infty} < \varepsilon$. This implies that $\left\| P_f\big|_{\Lambda} \right\|_{\infty} \leq \varepsilon$. With $\tilde{\varepsilon} = \varepsilon / \sum_{j=1}^{n} |\alpha_j|$ and $F = \{\psi_1, \ldots, \psi_n\}$, we obtain from Proposition 2.5 that Λ is an $(F, G, (6\pi + 2)\tilde{\varepsilon})$-sampling set. Given that $P_f \in A_F(G)$, we have as a consequence that

$$\|P_f\|_{\infty} \leq (6\pi + 2)\tilde{\varepsilon} \cdot \sum_{j=1}^{n} |\alpha_j| + \varepsilon \leq (6\pi + 3)\varepsilon.$$

Since $\|f - P_f\|_{\infty} \leq \varepsilon$, we deduce that $\|f\|_{\infty} < (6\pi + 4)\varepsilon$. We conclude that $f = 0$, for ε was arbitrary. □

In the case of almost periodic functions, every continuous function on $G^{\mathcal{AP}}$ coincides with an almost periodic function on G, and we obtain therefore that Λ is dense in $G^{\mathcal{AP}}$.

Corollary 2.7 *Let G be an Abelian topological group and $\Lambda \subseteq G$. Suppose that Λ is an (F, G, ε)-matching set for every finite subset $F \subseteq \widehat{G}$ and every $\varepsilon > 0$. Then Λ is a set of uniqueness for $\mathcal{AP}(G)$. In particular Λ is dense in $G^{\mathcal{AP}}$.*

Proof The only difference with Corollary 2.6 resides in the density statement. The Gelfand representation identifies $\mathcal{AP}(G)$ with $C(G^{\mathcal{AP}}, \mathbb{C})$. If Λ is not dense in $G^{\mathcal{AP}}$, by Urysohn's lemma there would be a nonconstant function $f : G^{\mathcal{AP}} \to \mathbb{C}$ that vanishes on Λ, which is impossible since f is determined by its values on Λ. □

To be able to cover simultaneously the cases of $\mathbb{R}$ and $\mathbb{Z}$, we state our next results in the context of locally compact groups with Haar measure.

Our next objective is to estimate the probability of selecting a set that is an $(F, \{a\}, \varepsilon)$-matching set for a fixed $a \in G$.

We start by introducing some notation.

Definition 2.8 For $I \subseteq G$, $n \in \mathbb{N}$, $a \in G$, and $\varepsilon > 0$, we define

$$\mathfrak{P}_{I,a,n,\varepsilon} = \left\{ F \subseteq C(G, \mathbb{T}) : |F| = n \text{ and } \lambda_G \left(N^{\triangleleft}(F, \varepsilon, a) \cap I \right) \geq \varepsilon^n \lambda_G(I) \right\}, \text{ and}$$
$$\mathfrak{P}_{I,n,\varepsilon} = \left\{ F \subseteq C(G, \mathbb{T}) : |F| = n \text{ and } \lambda_G \left(N^{\triangleleft}(F, \varepsilon, a) \cap I \right) \geq \varepsilon^n \lambda_G(I) \text{ for all } a \in G \right\}.$$

The sets $\mathfrak{P}_{I,a,n,\varepsilon}$ are formed by n-subsets of $C(G,\mathbb{T})$ whose elements are functions ϕ that map some point of I into $\phi(a)\cdot\mathcal{V}_\varepsilon$ with probability at least ε^n.

Definition 2.9 Let G be a locally compact Abelian group, $I\subseteq G$ and $a\in G$. For $\Delta\subseteq C(G,\mathbb{T})$, $n,\ell\in\mathbb{N}$, and $\varepsilon>0$, we define $\mathcal{A}_{a,\Delta,n,\ell,\varepsilon,I}$ to be the set of all ℓ-subsets $\Lambda\subseteq I$ such that, for every $F\subseteq\Delta$ with $F\in\mathfrak{P}_{I,a,n,\varepsilon}$,

$$\Lambda\cap N^{\lhd}(F,\varepsilon,a)\neq\varnothing.$$

The set $\mathcal{A}_{a,\Delta,n,\ell,\varepsilon,I}$ is made of all the ℓ-subsets of I containing some point that is mapped by all functions $\phi\in F\subseteq\Delta$ into $\phi(a)\cdot\mathcal{V}_\varepsilon$. In what follows, we regard the sets $\mathcal{A}_{a,\Delta,n,\ell,\varepsilon,I}$ as events in the probability space determined by the restriction of λ_{G^ℓ} to I^ℓ, see the remarks at the beginning of Sect. 1.1.

The sets $\mathfrak{P}_{I,a,n,\varepsilon}$ and $\mathcal{A}_{a,\Delta,n,\ell,\varepsilon,I}$ are tailored to facilitating these estimates, as the following lemma shows.

Lemma 2.10 *Let $I\subseteq G$ be a subset of positive Haar measure, and let $\Delta\subseteq C(G,\mathbb{T})$ be an N-subset. Consider as well $a\in G$, $\varepsilon>0$, and $n\in\mathbb{N}$. Then*

$$\mathbb{P}\left(\mathcal{A}^{\mathbf{c}}_{a,\Delta,n,\ell,\varepsilon,I}\right)\leq\binom{N}{n}\left(1-\varepsilon^n\right)^\ell\leq\left(\frac{Ne}{n}\right)^n\left(1-\varepsilon^n\right)^\ell.$$

Proof The second inequality is a well-known estimate of binomial coefficients. For the first inequality, we observe that

$$\mathcal{A}^{\mathbf{c}}_{a,\Delta,n,\ell,\varepsilon,I}=\bigcup\left\{\left[\left(N^{\lhd}(F,\varepsilon,a)\cap I\right)^{\mathbf{c}}\right]^\ell : F\subseteq\Delta,\ F\in\mathfrak{P}_{I,a,n,\varepsilon}\right\},$$

where $\mathbb{P}\Big(\left[\left(N^{\lhd}(F,\varepsilon,a)\cap I\right)^{\mathbf{c}}\right]^\ell\Big)\leq(1-\varepsilon^n)^\ell$ since $\lambda_G\big(\left(N^{\lhd}(F,\varepsilon,a)\cap I\right)^{\mathbf{c}}\big)\leq\lambda_G(I)(1-\varepsilon^n)$ for every $F\in\mathfrak{P}_{I,a,n,\varepsilon}$. The inequality follows because there are $\binom{N}{n}$ n-subsets of Δ. $\square$

In the following definition, we introduce the events $\mathcal{B}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*}$ which correspond to the determining sets we are constructing.

Definition 2.11 Let G be an LCA group, and let $I^*=(I_k)_{k\in\mathbb{N}}$ and $\Delta^*=(\Delta_k)_{k\in\mathbb{N}}$ be sequences of subsets of G and $C(G,\mathbb{T})$, respectively. Let as well a sequence $\ell^*=(\ell_k)_{k\in\mathbb{N}}$ of positive integers, $a\in G$, $n\in\mathbb{N}$, and $\varepsilon>0$ be given. We define the event

$$\mathcal{B}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*}=\left\{(\Lambda_k)_{k\in\mathbb{N}}\colon\text{There exists} N\in\mathbb{N}\text{ such that }\Lambda_k\in\mathcal{A}_{a,\Delta_k,n,\ell_k,\varepsilon,I_k}\text{ for }k\geq N\right\}.$$

We now estimate the probability of the events $\mathcal{B}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*}$ regarded as events in the probability space $\prod_k I_k^{\ell_k}$, where each factor is assumed to carry the probability measure induced by the restriction of the Haar measure on G^{ℓ_k}. It then follows that given a sequence of intervals $(I_k)_{k\in\mathbb{N}}$ and a sequence $(\Delta_k)_{k\in\mathbb{N}}$ of finite subsets of $C(G,\mathbb{T})$, a randomly chosen sequence of ℓ_k-subsets of the I_k's belongs to $\mathcal{B}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*}$ with probability one as long as the growth of ℓ_k is large enough (where "enough" is controlled by $\log k$ and the cardinality of the Δ_k's).

Lemma 2.12 *Consider the sequences*

(1) $I^* = (I_k)_{k\in\mathbb{N}}$, *with* $I_k \subseteq G$ *of nonzero Haar measure,*
(2) $\Delta^* = (\Delta_k)_{k\in\mathbb{N}}$, *with* $\Delta_k \subseteq C(G,\mathbb{T})$ *and* $L_k := |\Delta_k| < \infty$, *and*
(3) $\ell^* = (\ell_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$.

Let $n \in \mathbb{N}$ *and* $\varepsilon > 0$. *If there are* $\gamma > 0$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\frac{-n\log L_k}{\varepsilon^n} + \ell_k > \frac{-(1+\gamma)\log k}{\log(1-\varepsilon^n)} \tag{2.1}$$

for every $k \geq k_0$, *then* $\mathbb{P}\left(\mathcal{B}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*}\right) = 1$ *for every* $a \in G$.

Proof Since all indices except k are fixed throughout the proof, we denote by $\mathcal{A}_k$ the set $\mathcal{A}_{a,\Delta_k,n,\ell_k,\varepsilon,I_k}$ and for $k' \in \mathbb{N}$ we identify the set $\mathcal{A}_{k'}$ with the subset $\prod_k \mathcal{X}_k \subseteq \prod_k I_k^{\ell_k}$ defined by $\mathcal{X}_{k'} := \mathcal{A}_{k'}$ and $\mathcal{X}_k := I_k^{\ell_k}$ for $k \neq k'$. Note that the probabilities of the event $\mathcal{A}_{k'}$ in the probability space $\prod_k I_k^{\ell_k}$ and in $I_{k'}^{\ell_{k'}}$ coincide.

Using this identification, we have that

$$\mathcal{B}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*} = \bigcup_{N\in\mathbb{N}} \bigcap_{k\geq N} \mathcal{A}_k = \limsup \mathcal{A}_k.$$

We now see that $\mathbb{P}(\mathcal{B}^{\mathbf{c}}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*}) = 0$. In fact, it follows from Lemma 2.10 that

$$\sum_{k\geq k_0} \mathbb{P}\left(\mathcal{A}_k^{\mathbf{c}}\right) \leq \sum_{k\geq k_0} \left(\frac{L_k e}{n}\right)^n (1-\varepsilon^n)^{\ell_k} = \left(\frac{e}{n}\right)^n \sum_{k\geq k_0} (1-\varepsilon^n)^{\frac{n\log L_k}{\log(1-\varepsilon^n)}+\ell_k}.$$

Since $\log(1-\varepsilon^n) < -\varepsilon^n$, we obtain $\frac{n\log L_k}{\log(1-\varepsilon^n)} + \ell_k > \frac{-n\log L_k}{\varepsilon^n} + \ell_k$, and by (2.1) we get

$$\sum_{k\geq k_0} \mathbb{P}\left(\mathcal{A}_k^{\mathbf{c}}\right) \leq \left(\frac{e}{n}\right)^n \sum_{k\geq k_0} (1-\varepsilon^n)^{\frac{-n\log L_k}{\varepsilon^n}+\ell_k} \leq \sum_{k\geq k_0} \left[(1-\varepsilon^n)^{\frac{-(1+\gamma)}{\log(1-\varepsilon^n)}}\right]^{\log k},$$

which is a convergent series of the form $\sum_{k\geq k_0} x^{\log k}$ with $|x| < \frac{1}{e}$. The Borel–Cantelli lemma then implies

$$\mathbb{P}(\mathcal{B}^{\mathbf{c}}_{a,\Delta^*,n,\ell^*,\varepsilon,I^*}) = \mathbb{P}\left((\limsup \mathcal{A}_k)^{\mathbf{c}}\right) = 0,$$

as required. □

3 Matching Intervals and Characters

In this section $G = \mathbb{R}$ or $G = \mathbb{Z}$. Recall that for $p \in \mathbb{R}[x]$, $\psi_p(t) = e^{2\pi i p(t)}$, and for $\tau \in \mathbb{R}$, $\chi_\tau(x) = e^{2\pi i \tau x}$.

Definition 3.1 ([8, Definition 2.74]) Let $\mathbf{x}\colon [0,+\infty) \to \mathbb{R}^n$ be a continuous function. The *continuous discrepancy of* $\mathbf{x}$ *in* $[0, T]$ is defined by

$$D_T(\mathbf{x}) = \sup_{\mathcal{V}\subseteq\mathbb{T}^n} \left| \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{V}}\left(e^{2\pi i\mathbf{x}(t)}\right) dt - \lambda_{\mathbb{T}^n}(\mathcal{V}) \right|,$$

where $\mathbf{1}_{\mathcal{V}}$ denotes the characteristic function of the set $\mathcal{V}$, $e^{2\pi i\mathbf{x}(t)}$ stands for the vector $\left(e^{2\pi i x_1(t)}, \dots, e^{2\pi i x_n(t)}\right) \in \mathbb{T}^n$, and the supremum is taken over all rectangles in $\mathbb{T}^n$ with sides parallel to the axes.

Definition 3.2 ([8, Definition 2.70 and Theorem 2.75]) A function $\mathbf{x} : [0,+\infty) \to \mathbb{R}^n$ is *continuously well distributed modulo 1* if it is continuous and $\lim_{T\to\infty} D_T(\mathbf{x}(t+\tau)) = 0$ uniformly in τ.

We propose the following definition to make our notation lighter.

Definition 3.3 The polynomials $\{p_1, \dots, p_n\} \subseteq \mathbb{R}[x]$ are *strongly linearly independent over* $\mathbb{Q}$ if for each nonzero $(h_1, \dots, h_n) \in \mathbb{Z}^n$ the polynomial $\sum_{j=1}^n h_j p_j$ is nonconstant.

The functions $\{\psi_{p_1}, \dots, \psi_{p_n}\}$ are *strongly linearly independent over* $\mathbb{Q}$ if $\{p_1, \dots, p_n\} \subseteq \mathbb{R}[x]$ are strongly linearly independent over $\mathbb{Q}$.

Let H be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$, and let C_H denote the constant polynomials with values in H. The polynomials $\{p_1, \dots, p_n\} \subseteq \mathbb{R}[x]$ are strongly linearly independent over $\mathbb{Q}$ if and only if the set $\{p_1, \dots, p_n\} \cup C_H$ is linearly independent over $\mathbb{Q}$.

Theorem 3.4 ([8, Corollary of Theorems 2.73 and 2.79]) *If* $\{p_1, \dots, p_n\} \subseteq \mathbb{R}[x]$ *are strongly linearly independent over* $\mathbb{Q}$*, then the function* $\mathbf{x}\colon [0,+\infty) \to \mathbb{R}^n$ *defined by* $\mathbf{x}(t) = (p_1(t), \dots, p_n(t))$ *is continuously well distributed modulo 1.*

Proof By Weyl's criterion for continuous well-distribution [8, Theorem 2.73], it suffices to prove that for every nonzero $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{Z}^n$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{2\pi i\mathbf{h}\cdot x(t+\tau)}\, dt = 0, \tag{3.1}$$

uniformly in $\tau \geq 0$.

Let $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{Z}^n$ be nonzero. Since the polynomial $q_h(t) = \sum_{j=1}^n h_j p_j(t)$ is nonconstant, there exist $t_0 \in \mathbb{R}$ and $C > 0$ such that $|q_h(t)| \geq C$ and $q_h''(t)$ has constant sign for every $t \geq t_0$. From [8, Theorem 2.79] it then follows that q_h is continuously well distributed, and Weyl's criterion applied to q_h shows that (3.1) holds. □

If F is a family of $\mathbb{T}$-valued functions, we are trying to estimate how long an interval I should be for F to be as likely as expected to send some element of I into a fixed neighbourhood of $\mathbb{T}$. We next see, as a consequence of Theorem 3.4, that for polynomial phase functions generated by strongly linearly independent polynomials, this happens as soon as the length of I exceeds a bound that depends only on the cardinality of the family and the size of the neighbourhood.

Theorem 3.5 *Let $F = \{\psi_{p_1}, \dots, \psi_{p_n}\}$ where $\{p_1, \dots, p_n\} \subseteq \mathbb{R}[x]$ are strongly linearly independent over $\mathbb{Q}$, and let $\gamma > 0$. Then there exists $L(F, \gamma) > 0$ such that for every interval $I \subseteq \mathbb{R}$ with $\lambda_{\mathbb{R}}(I) \geq L(F, \gamma)$ and every $a \in \mathbb{R}$,*

$$\lambda_{\mathbb{R}}\left(N^{\lhd}(F, \delta, a) \cap I\right) \geq \left((2\delta)^n - \gamma\right) \lambda_{\mathbb{R}}(I),$$

for every $\delta > 0$ with $(2\delta)^n - \gamma > 0$. In particular, $F \in \mathfrak{P}_{I,n,\delta}$ if $\lambda_{\mathbb{R}}(I) \geq L(F, \delta^n(2^n - 1))$.

Proof Fix $a \in \mathbb{R}$ and $\gamma > 0$. Define $\mathbf{x}_F\colon [0, +\infty) \to \mathbb{R}^n$ by $\mathbf{x}_F(t) = (p_1(t), \dots, p_n(t))$ and put, for each $\delta > 0$, $\mathcal{V}_{\delta,a} = \psi_{p_1}(a) \cdot \mathcal{V}_\delta \times \cdots \times \psi_{p_n}(a) \cdot \mathcal{V}_\delta \subseteq \mathbb{T}^n$. By Theorem 3.4 the function $\mathbf{x}_F(t)$ is continuously well distributed. There is accordingly $L(F, \gamma) > 0$ such that $T \geq L(F, \gamma)$ implies $D_T(\mathbf{x}(t + \tau)) \leq \gamma$ for every τ, i.e.,

$$\frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{V}_{\delta,a}}\left(e^{2\pi i \mathbf{x}_F(t+\tau)}\right) dt \geq (2\delta)^n - \gamma. \tag{3.2}$$

Let now $I = [\tau_0, \tau_0 + L] \subseteq \mathbb{R}$ be an arbitrary interval of length $L \geq L(F, \gamma)$. Taking into account the definition of $\mathcal{V}_{\delta,a}$, inequality (3.2) applied to $\tau = \tau_0$ and $T = L$ implies that

$$\begin{aligned} L \cdot \left((2\delta)^n - \gamma\right) &\leq \lambda_{\mathbb{R}}\left(\left\{t \in [0, L]\colon \psi_{p_j}(t + \tau_0) \in \psi_{p_j}(a) \cdot \mathcal{V}_\delta \text{ for } j = 1, \dots n\right\}\right) \\ &= \lambda_{\mathbb{R}}\left(N^{\lhd}(F, \delta, a) \cap I\right). \end{aligned}$$

□

The same argument of Theorem 3.5 with well distributed sequences instead of continuously well distributed functions can be used for $G = \mathbb{Z}$.

Corollary 3.6 *Let $F = \{\psi_{p_1}, \dots, \psi_{p_n}\}$ where $\{p_1, \dots, p_n\} \subseteq \mathbb{R}[x]$ have coefficients in $[0, 1)$ and are strongly linearly independent over $\mathbb{Q}$, and let $\gamma > 0$. Then there is $L(F, \gamma) > 0$ such that, for every interval $I \subseteq \mathbb{Z}$ with $|I| \geq L(F, \gamma)$ and every $a \in \mathbb{Z}$,*

$$|N^{\lhd}(F, \delta, a) \cap I| \geq \left((2\delta)^n - \gamma\right) |I|,$$

for every $\delta > 0$ with $(2\delta)^n - \gamma > 0$. In particular, $F \in \mathfrak{P}_{I,n,\delta}$ if $|I| \geq L(F, \delta^n(2^n - 1))$.

To close this section, we consider sets of characters instead of sets of more general continuous $\mathbb{T}$-valued functions. In this case $F \in \mathfrak{P}_{I,a,n,\varepsilon}$ if and only if $F \in \mathfrak{P}_{I,n,\varepsilon}$, and therefore we localize our arguments at the identity. For some special sets of characters F, we can actually find a more concrete bound for Theorem 3.5.

Theorem 3.7 *Let* $0 < \frac{q_1}{p_1} < \cdots < \frac{q_n}{p_n} = 1$ *be a finite sequence of rationals and* $\varepsilon \in \left(0, \frac{1}{2}\right)$. *If* $I \subseteq \mathbb{R}$ *is an interval with* $\lambda_{\mathbb{R}}(I) \geq p_1 \cdots p_{n-1}$, *then* $F := \{\chi_{\frac{q_1}{p_1}}, \ldots, \chi_{\frac{q_n}{p_n}}\} \in \mathfrak{P}_{I,n,\varepsilon}$.

Proof Let $N := p_1 \cdots p_{n-1}$. We can assume that the fractions $\frac{q_1}{p_1}, \ldots, \frac{q_{n-1}}{p_{n-1}}$ are irreducible. If they are not, we work with the simplified fractions and obtain a smaller N that works.

We first assume $\lambda_{\mathbb{R}}(I) = N$ and define $J_k := \{0, 1, \ldots, \lfloor p_k\varepsilon \rfloor, p_k - \lfloor p_k\varepsilon \rfloor, \ldots, p_k - 1\}$ for each $k = 1, \ldots, n-1$. We claim that

(1) for each $\mathsf{l} \in \prod_{k=1}^{n-1} J_k$ there exists $z_{\mathsf{l}} \in \mathbb{Z}$ such that either $[z_{\mathsf{l}}, z_{\mathsf{l}} + \varepsilon]$ or $[z_{\mathsf{l}} - \varepsilon, z_{\mathsf{l}}]$ is contained in $N^{\lhd}(F, \varepsilon) \cap I$,
(2) for $z_{\mathbf{0}}$ we have $[z_{\mathbf{0}} - \varepsilon, z_{\mathbf{0}} + \varepsilon] \subseteq N^{\lhd}(F, \varepsilon) \cap I$, and
(3) the integers z_{l} are all different.

Indeed, for fixed $j_{n-1} \in J_{n-1}$ we consider the set $L_{n-1,j_{n-1}}$ consisting of those integers in I that are mapped to $e^{2\pi i j_{n-1}/p_{n-1}}$ by $\chi_{\frac{q_{n-1}}{p_{n-1}}}$; i.e.,

$$L_{n-1,j_{n-1}} = \left\{z \in \mathbb{Z} \cap I \colon \frac{q_{n-1}}{p_{n-1}} \cdot z = \frac{j_{n-1}}{p_{n-1}} + \ell \text{ for some } \ell \in \mathbb{Z}\right\}.$$

Since (the class of) q_{n-1} is a generator of the cyclic group $\mathbb{Z}/p_{n-1}\mathbb{Z}$, the set $L_{n-1,j_{n-1}}$ contains precisely N/p_{n-1} integers with a distance of p_{n-1} between consecutive ones. For $z \in L_{n-1,j_{n-1}}$, either $[z, z+\varepsilon]$ or $[z-\varepsilon, z]$ is contained in I and sent into $\mathcal{V}_\varepsilon$ by both χ_1 and $\chi_{\frac{q_{n-1}}{p_{n-1}}}$. For $z \in L_{n-1,0}$, both characters map $[z-\varepsilon, z+\varepsilon]$ into $\mathcal{V}_\varepsilon$.

Next we fix $j_{n-2} \in J_{n-2}$ and consider the set $L_{n-2,j_{n-2}}$ of those elements of $L_{n-1,j_{n-1}}$ that are sent to $e^{2\pi i j_{n-2}/p_{n-2}}$ by $\chi_{\frac{q_{n-2}}{p_{n-2}}}$; i.e.,

$$L_{n-2,j_{n-2}} = \left\{z \in L_{n-1,j_{n-1}} \colon \frac{q_{n-2}}{p_{n-2}} \cdot z = \frac{j_{n-2}}{p_{n-2}} + \ell \text{ for some } \ell \in \mathbb{Z}\right\}.$$

As before, exactly $N/(p_{n-1}p_{n-2})$ elements of $L_{n-1,j_{n-1}}$ belong to $L_{n-2,j_{n-2}}$, and the distance between any two consecutives is $p_{n-1}p_{n-2}$. For $z \in L_{n-2,j_{n-2}}$, either $[z, z+\varepsilon]$ or $[z-\varepsilon, z]$ is sent into $\mathcal{V}_\varepsilon$ by χ_1, $\chi_{\frac{q_{n-1}}{p_{n-1}}}$ and $\chi_{\frac{q_{n-2}}{p_{n-2}}}$. For $z \in L_{n-2,0}$, these characters map $[z-\varepsilon, z+\varepsilon]$ into $\mathcal{V}_\varepsilon$.

After $(n-1)$ steps the components of $\mathsf{l} \in \prod_{k=1}^{n-1} J_k$ have been fixed, the set L_{1,j_1} contains precisely one integer, say $z_{\mathsf{l}} \in \bigcap_{k=1}^{n-1} L_{k,j_k}$, and $[z_{\mathsf{l}}, z_{\mathsf{l}}+\varepsilon]$ or $[z_{\mathsf{l}}-\varepsilon, z_{\mathsf{l}}]$ is contained in $N^{\lhd}(F, \varepsilon) \cap I$. Since each l produces a different z_{l}, our claim is proved.

Since $|J_k| = 2\lfloor p_k\varepsilon \rfloor + 1$ for each k, and the interval around z_{l} have length at least ε, the intervals constructed in the previous claim have a total length of

$$\varepsilon \cdot \left[\prod_{k=1}^{n-1} (2\lfloor p_k\varepsilon \rfloor + 1) + 1\right] \geq 2\varepsilon^n N. \tag{3.3}$$

In fact, since $2\lfloor n\varepsilon\rfloor + 1 \geq n\varepsilon$ for every $n \in \mathbb{N}$, in case $p_i\varepsilon, p_j\varepsilon \geq \frac{1}{2-\sqrt{2}}$ for some $i \neq j$ then $(2\lfloor p_i\varepsilon\rfloor+1)(2\lfloor p_j\varepsilon\rfloor+1) \geq (2p_i\varepsilon-1)(2p_j\varepsilon-1) \geq (\sqrt{2}p_i\varepsilon)(\sqrt{2}p_j\varepsilon) = 2\varepsilon^2 p_i p_j$, and (3.3) follows. On the other hand, if $1 \leq p_i\varepsilon \leq \frac{1}{2-\sqrt{2}}$ for some i, then $3 = (2\lfloor p_i\varepsilon\rfloor+1) \geq 2p_i\varepsilon$, and inequality (3.3) holds. The only remaining case is when $p_i\varepsilon < 1$ for every i except at most one i_0. In that case, from $2\lfloor p_{i_0}\varepsilon\rfloor + 1 \geq 2p_{i_0}\varepsilon - 1$ we also obtain that (3.3) holds since its left-hand side is bounded below by

$$\varepsilon \cdot [(2\lfloor p_{i_0}\varepsilon\rfloor + 1) + 1] \geq \varepsilon \cdot (p_1\varepsilon)\cdots(2p_{i_0}\varepsilon)\cdots(p_{n-1}\varepsilon) = 2\varepsilon^n N.$$

We have thus shown that $\lambda_{\mathbb{R}}\left(N^{\lhd}(F,\varepsilon) \cap I\right) \geq 2\varepsilon^n\lambda_{\mathbb{R}}(I)$ when $\lambda_{\mathbb{R}}(I) = N$, as desired. In case $\lambda_{\mathbb{R}}(I) > N$, there exist $j \in \mathbb{N}$ and $\delta \in [0, N)$ such that $\lambda_{\mathbb{R}}(I) = jN + \delta$. Therefore, the interval I can be split into j-many subintervals of length N and another one of length δ. In each of the intervals of length N we can argue as above and find a family of subintervals of $N^{\lhd}(F,\varepsilon) \cap I$ whose accumulated length is $2N\varepsilon$. We then obtain that

$$\lambda_{\mathbb{R}}\left(N^{\lhd}(F,\varepsilon) \cap I\right) \geq 2jN\varepsilon^n \geq 2\varepsilon^n\left(1 - \frac{1}{j+1}\right)(jN+\delta) \geq \varepsilon^n\lambda_{\mathbb{R}}(I).$$

□

Corollary 3.8 *Let* $0 < \frac{q_1}{p_1} < \cdots < \frac{q_n}{p_n}$ *be a finite sequence of rationals and* $\varepsilon \in \left(0, \frac{1}{2}\right)$. *If* $I \subseteq \mathbb{R}$ *is an interval with* $\lambda_{\mathbb{R}}(I) \geq p_1\cdots p_n q_n^{n-2}$, *then* $F := \{\chi_{\frac{q_1}{p_1}}, \ldots, \chi_{\frac{q_n}{p_n}}\} \in \mathfrak{P}_{I,n,\varepsilon}$.

Proof From $\lambda_{\mathbb{R}}(I) \geq p_1\cdots p_n q_n^{n-2}$ we get $\lambda_{\mathbb{R}}(\frac{q_n}{p_n}I) \geq (q_n p_1)\cdots(q_n p_{n-1})$, and Theorem 3.7 then implies $\frac{p_n}{q_n}F := \{\chi_{\frac{p_n q_1}{q_n p_1}}, \ldots, \chi_{\frac{p_n q_{n-1}}{q_n p_{n-1}}}, \chi_1\} \in \mathfrak{P}_{\frac{q_n}{p_n}I,n,\varepsilon}$. The result follows because $N^{\lhd}(\frac{p_n}{q_n}F,\varepsilon) \cap \frac{q_n}{p_n}I = \frac{q_n}{p_n}\left(N^{\lhd}(F,\varepsilon) \cap I\right)$ implies $\frac{p_n}{q_n}F \in \mathfrak{P}_{\frac{q_n}{p_n}I,n,\varepsilon}$ if and only if $F \in \mathfrak{P}_{I,n,\varepsilon}$. □

The interval can be shortened in the presence of certain algebraic relations in F.

Corollary 3.9 *Let* $0 < \frac{q_1}{p} < \cdots < \frac{q_n}{p}$ *be a finite sequence of rationals and* $\varepsilon \in \left(0, \frac{1}{2}\right)$. *If* $I \subseteq \mathbb{R}$ *is an interval with* $\lambda_{\mathbb{R}}(I) \geq pq_n^{n-2}$, *then* $F := \{\chi_{\frac{q_1}{p}}, \ldots, \chi_{\frac{q_n}{p}}\} \in \mathfrak{P}_{I,n,\varepsilon}$.

Proof From $\lambda_{\mathbb{R}}(I) \geq pq_n^{n-2}$ we obtain $\lambda_{\mathbb{R}}(\frac{q_n}{p}I) \geq q_n^{n-1}$ and Theorem 3.7 then asserts $\frac{p}{q_n}F = \{\chi_{\frac{q_1}{q_n}}, \chi_{\frac{q_2}{q_n}}, \ldots, \chi_1\} \in \mathfrak{P}_{\frac{q_n}{p}I,n,\varepsilon}$, i.e., $F \in \mathfrak{P}_{I,n,\varepsilon}$. □

A considerably shorter interval is needed when F is sparse enough.

Corollary 3.10 *Let* $F := \{\chi_{\tau_1}, \ldots, \chi_{\tau_n}\}$ *be such that* $\frac{\tau_{j+1}}{\tau_j} > \frac{1}{2\varepsilon}$, $j = 1, \ldots, n-1$, *for some* $\varepsilon \in \left(0, \frac{1}{2}\right)$. *If* $I \subseteq \mathbb{R}$ *is an interval with* $\lambda_{\mathbb{R}}(I) \geq \frac{1}{\tau_1}$, *then* $F \in \mathfrak{P}_{I,n,\varepsilon}$.

Proof If $\lambda_{\mathbb{R}}(I) = \frac{1}{\tau_1}$, there exists $I_1 \subseteq I$ such that $\lambda_{\mathbb{R}}(I_1) = \frac{2\varepsilon}{\tau_1}$ and $\chi_{\tau_1}[I_1] = \mathcal{V}_\varepsilon$. Then $\chi_{\tau_2}[I_1] = \mathbb{T}$, and there exists $I_2 \subseteq I_1$ such that $\lambda_{\mathbb{R}}(I_2) = 2\varepsilon \cdot \frac{2\varepsilon}{\tau_1}$ and $\chi_{\tau_2}[I_2] = \mathcal{V}_\varepsilon$.

At the nth step we find $I_n \subseteq I_{n-1}$ such that $\lambda_{\mathbb{R}}(I_n) = \frac{(2\varepsilon)^n}{\tau_1}$ and $\chi_{\tau_n}[I_n] \subseteq \mathcal{V}_\varepsilon$. It follows that $I_n \subseteq N^{\lhd}(F, \varepsilon) \cap I$ and $F \in \mathfrak{P}_{I,n,\varepsilon}$. Finally, for $\lambda_{\mathbb{R}}(I) \geq \frac{1}{\tau_1}$ there exist $j \in \mathbb{N}$ and $\delta \in [0, \frac{1}{\tau_1})$ such that $\lambda_{\mathbb{R}}(I) = \frac{j}{\tau_1} + \delta$. Split I into j-many subintervals of length $\frac{1}{\tau_1}$ plus another one of length δ. By the above argument, each of the former intervals contains a subinterval of length $\frac{(2\varepsilon)^n}{\tau_1}$ and the proof then goes as in Theorem 3.7. ◻

4 Random Bohr Dense Subsets

In this section we combine the results of Sects. 2 and 3 in order to show that if enough points are randomly chosen from each element of a sequence of sufficiently large intervals, then almost surely we obtain a Bohr-dense set. The estimates in Sect. 3 are first used to find criteria for a collection of finite choices in a sequence of long enough intervals of real numbers to be an $(F, \mathbb{R}, \varepsilon)$-matching set for every m-set F (with $m \in \mathbb{N}$ fixed) of strongly linearly independent polynomial phase functions. The estimates of Sect. 2 are then used to see that these criteria are almost surely met. This yields sets that are $(F, \mathbb{R}, \varepsilon)$-matching for every finite set of degree n polynomial phase functions and every $\varepsilon > 0$, that is, sets of uniqueness for $\mathcal{AP}_{\mathfrak{C}_n}$.

Lemma 4.1 *Let $I^* = (I_k)_{k\in\mathbb{N}}$ be a sequence of intervals $I_k = [a_k, a_k + b_k] \subseteq \mathbb{R}$ with $\limsup(b_k)_{k\in\mathbb{N}} = +\infty$. For each $k \in \mathbb{N}$, put $t_k := \max(|a_k|, |a_k + b_k|)$ and let Δ_k be a finite subset of $C(\mathbb{R}, \mathbb{T})$ whose restrictions to $[-t_k, t_k]$ are ε_k-dense in the restriction of $\mathfrak{C}_n$ to $[-t_k, t_k]$, where $\varepsilon_k \to 0$ as $k \to \infty$. Let $\Delta^* = (\Delta_k)_{k\in\mathbb{N}}$, and let $\ell^* = (\ell_k)_{k\in\mathbb{N}}$ be a sequence of positive integers. If $(d_s)_{s\in\mathbb{N}} \subseteq \mathbb{R}$ is a dense subset and for some $m \in \mathbb{Z}$ and $\varepsilon > 0$ we have $(\Lambda_k)_{k\in\mathbb{N}} \in \bigcap_{s\in\mathbb{N}} \mathcal{B}_{d_s,\Delta^*,m,\ell^*,\varepsilon,I^*}$, then $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is an $(F, \mathbb{R}, \varepsilon)$-matching set for every m-subset F of $\mathfrak{C}_n$ strongly linearly independent over $\mathbb{Q}$.*

Proof Let $F = \{\psi_{p_1}, \ldots \psi_{p_m}\} \subseteq \mathfrak{C}_n$ with $\{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x]$ strongly linearly independent over $\mathbb{Q}$. Fix $x_0 \in \mathbb{R}$ and consider $N \in \mathbb{N}$ and $s_0 \in \mathbb{N}$ with $\frac{3}{N} < \varepsilon$ and

$$d_a\big(\psi_{p_j}(d_{s_0}), \psi_{p_j}(x_0)\big) < \frac{1}{N}, \text{ for } j = 1, \ldots, m. \tag{4.1}$$

Since $(\Lambda_k)_k \in \mathcal{B}_{d_{s_0},\Delta^*,m,\ell^*,\frac{1}{N},I^*}$, there is k_0 such that for every $k \geq k_0$,

$$\Lambda_k \in \mathcal{A}_{d_{s_0},\Delta_k,m,\ell_k,\frac{1}{N},I_k}. \tag{4.2}$$

Applying Theorem 3.5 to $\delta, \gamma > 0$ such that $\delta < \frac{1}{N} < 2\delta$ and $(2\delta)^m - \gamma > N^{-m}$, we can find $k \geq k_0$ large enough to satisfy (4.2), $\varepsilon_k < \frac{1}{2}(\frac{1}{N} - \delta)$, $b_k > L(F, \gamma)$ and $d_{s_0} \in [-t_k, t_k]$. Since Δ_k is ε_k-dense in the restriction of $\mathfrak{C}_n$ to $[-t_k, t_k]$, for every $j = 1, \ldots, m$ there exists $\phi_j \in \Delta_k$ such that for every $t \in [-t_k, t_k]$,

$$d_a\left(\phi_j(t), \psi_{p_j}(t)\right) < \varepsilon_k < \frac{1}{2}\left(\frac{1}{N} - \delta\right). \tag{4.3}$$

From (4.1) and (4.3) it follows that, for $t \in N^{\lhd}\left(F, \delta, d_{s_0}\right) \cap I_k$, we have

$$d_a\left(\phi_j(t), \phi_j(d_{s_0})\right) \leq d_a\left(\phi_j(t), \psi_{p_j}(t)\right) + d_a\left(\psi_{p_j}(t), \psi_{p_j}(d_{s_0})\right) + d_a\left(\psi_{p_j}(d_{s_0}), \phi_j(d_{s_0})\right) < \frac{1}{N},$$

thus $N^{\lhd}\left(F, \delta, d_{s_0}\right) \cap I_k \subseteq N^{\lhd}(\tilde{F}, \frac{1}{N}, d_{s_0}) \cap I_k$, where $\tilde{F} := \{\phi_1, \ldots, \phi_m\}$. Since $b_k > L(F, \gamma)$ implies $\lambda_{\mathbb{R}}(N^{\lhd}(F, \delta, d_{s_0}) \cap I_k) \geq N^{-m}\lambda_{\mathbb{R}}(I_k)$, we conclude that $\tilde{F} \in \mathfrak{P}_{I_k, d_{s_0}, m, \frac{1}{N}}$, which together with (4.2) implies $\Lambda_k \cap N^{\lhd}(\tilde{F}, \frac{1}{N}, d_{s_0}) \neq \varnothing$. From (4.1) and (4.3) we then obtain that for $y \in \Lambda_k \cap N^{\lhd}(\tilde{F}, \frac{1}{N}, d_{s_0})$,

$$d_a\left(\psi_{p_j}(y), \psi_{p_j}(x_0)\right) \leq d_a\left(\psi_{p_j}(y), \phi_j(y)\right) + d_a\left(\phi_j(y), \phi_j(d_{s_0})\right) + d_a\left(\phi_j(d_{s_0}), \psi_{p_j}(d_{s_0})\right) + d_a\left(\psi_{p_j}(d_{s_0}), \psi_{p_j}(x_0)\right) < \frac{3}{N},$$

i.e., $\psi_{p_j}(y) \in \psi_{p_j}(x_0) \cdot \mathcal{V}_{\frac{3}{N}} \subseteq \psi_{p_j}(x_0) \cdot \mathcal{V}_{\varepsilon}$. In conclusion, Λ is an $(F, \mathbb{R}, \varepsilon)$-matching set. $\square$

Theorem 4.2 *Let $I^* = (I_k)_{k\in\mathbb{N}}$ be a sequence of intervals $I_k = [a_k, a_k + b_k] \subseteq \mathbb{R}$ with $\limsup(b_k)_{k\in\mathbb{N}} = +\infty$ and $t_k := \max(|a_k|, |a_k + b_k|) \geq k$ for every $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $\Lambda_k \subseteq I_k$ be a random subset with $|\Lambda_k| = \ell_k$ and $\ell_k \neq O(\log t_k)$. Then, for any fixed $n \in \mathbb{N}$, the set $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is, almost surely, a set of uniqueness for $\mathcal{AP}_{\mathfrak{C}_n}$.*

Proof We divide this proof into Steps. We first determine that a certain number of conditions in the selection of the Λ_k's are satisfied with probability one, and then we show that, when the sets Λ_k meet these conditions, then the set $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is a set of uniqueness for $\mathcal{AP}_{\mathfrak{C}_n}$.

Step 1: *Exhibiting an event $\mathcal{B}$ of probability one.*
Fix $n \in \mathbb{N}$. Since $\ell_k \neq O\big(\log t_k\big)$, the set

$$\mathcal{K}_{N,m} = \left\{k \in \mathbb{N}\colon \ell_k \geq (n+1)mN^m \log 2 + \left[mN^{m\,\frac{(n+4)(n+1)}{2}} - \frac{2}{\log(1-N^{-m})}\right]\log t_k\right\}$$

is infinite for each $N, m \in \mathbb{N}$. For each $k \in \mathbb{N}$ define

$$\widetilde{\Delta_k} = \left\{\sum_{r=0}^{n} \frac{j_r}{t_k^{r+1}} x^r \colon -\lfloor t_k^{r+2}\rfloor \leq j_r < \lfloor t_k{}^{r+2}\rfloor,\, j_r \in \mathbb{Z}, r = 0, \ldots, n\right\} \subseteq \mathbb{R}_n[x]$$

and $\Delta_k = \left\{\psi_p\colon p \in \widetilde{\Delta_k}\right\}$. Observe that $|\Delta_k| = \prod_{r=0}^{n}(2\lfloor t_k^{(r+2)}\rfloor) \leq 2^{n+1} t_k^{\frac{(n+1)(n+4)}{2}}$. Fix $m, N \in \mathbb{N}$. If $k \in \mathcal{K}_{N,m}$, then

$$\begin{aligned}\ell_k &\geq (n+1)mN^m \log 2 + \left[mN^m \frac{(n+4)(n+1)}{2} - \frac{2}{\log\left(1-N^{-m}\right)} \right] \log t_k \\ &\geq mN^m \log |\Delta_k| - \frac{2}{\log\left(1-N^{-m}\right)} \log k,\end{aligned}$$

and Lemma 2.12 applied to the sequences $I^*_{N,m} = (I_k)_{k\in\mathcal{K}_{N,m}}$, $\Delta^*_{N,m} = (\Delta_k)_{k\in\mathcal{K}_{N,m}}$, and $\ell^*_{N,m} = (\ell_k)_{k\in\mathcal{K}_{N,m}}$ with $\varepsilon = \frac{1}{N}$ and with n replaced by m, yields then that for every $q \in \mathbb{R}$ the event $\mathcal{B}_{q,\Delta^*_{N,m},m,\ell^*_{N,m},\frac{1}{N},I^*_{N,m}}$ occurs with probability one. Therefore, if $(d_s)_{s\in\mathbb{N}} \subseteq \mathbb{R}$ denotes a countable dense subset, the event

$$\mathcal{B} = \bigcap_{s\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcap_{N\in\mathbb{N}} \mathcal{B}_{d_s,\Delta^*_{N,m},m,\ell^*_{N,m},\frac{1}{N},I^*_{N,m}}$$

also occurs with probability one.

Step 2: The set Δ_k *is* $\frac{n+1}{t_k}$*-dense in the restrictions of* $\mathfrak{C}_n$ *to* $[-t_k, t_k]$.
Consider any element $f(x) = e^{2\pi i p(x)}$ of $\mathfrak{C}_n$ with $p(x) = \sum_{r=0}^n a_r x^r \in \mathbb{R}[x]$. For each $r \in \{0, \dots, n\}$ find $j_r \in \mathbb{Z}$ such that $\left|a_r - \frac{j_r}{t_k^{r+1}}\right| \leq \frac{1}{t_k^{r+1}}$. Then, the polynomial $q(x) = \sum_{r=0}^n \frac{j_r}{t_k^{r+1}} x^r$ is such that $h(x) = e^{2\pi i q(x)} \in \Delta_k$, and for every $x \in [-t_k, t_k]$ we obtain

$$\begin{aligned}d_a(h(x), f(x)) \leq |p(x) - q(x)| &\leq \sum_{r=0}^n \left| a_r - \frac{j_r}{t_k^{r+1}} \right| \cdot |x|^r \\ &\leq \sum_{r=0}^n \frac{|x|^r}{t_k^{r+1}} \leq \sum_{r=0}^n \frac{t_k^r}{t_k^{r+1}} = \frac{n+1}{t_k}.\end{aligned}$$

Step 3: *If* $(\Lambda_k)_{k\in\mathbb{N}} \in \mathcal{B}$, *then* $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ *is* $(F, \mathbb{R}, \varepsilon)$*-matching for every* $F \subseteq \mathfrak{C}_n$ *induced by a finite family of polynomials that is strongly linearly independent over* $\mathbb{Q}$ *and every* $\varepsilon > 0$.

For such a set Λ, fix $\varepsilon > 0$ and let $F = \{\psi_{p_1}, \dots, \psi_{p_s}\}$ with $\{p_1, \dots, p_s\} \subseteq \mathbb{R}_n[x]$, strongly linearly independent over $\mathbb{Q}$. Fix $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. By our Step 2 above, Lemma 4.1 applies to the sequences $I^*_{N,m}$, $\Delta^*_{N,m}$, and $\ell^*_{N,m}$ and shows that Λ is an $(F, \mathbb{R}, \varepsilon)$ matching set.

Step 4: *If* $(\Lambda_k)_{k\in\mathbb{N}} \in \mathcal{B}$, *then* $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is $(F, \mathbb{R}, \varepsilon)$*-matching for every finite set* $F \subseteq \mathfrak{C}_n$ *and every* $\varepsilon > 0$.

Fix $\varepsilon > 0$, and let $F = \{\psi_{p_1}, \dots, \psi_{p_s}\}$ with $\{p_1, \dots, p_s\} \subseteq \mathbb{R}_n[x]$. We may assume that $\{p_1, \dots, p_m\}$ are strongly linearly independent over $\mathbb{Q}$ and that, for each $j = m+1, \dots, s$, there is a constant $C_j \in \mathbb{R}$ and there are integers Q and Z_{ij}, $1 \leq i \leq m$, with

$$p_j = C_j - \sum_{i=1}^m \frac{Z_{ij}}{Q} p_i.$$

Let us define $M = \max\{\sum_{i=1}^{m} |Z_{ij}| : j \in [m+1, N]\}$ and $\tilde{\varepsilon} = \varepsilon/M$. Fix $N \in \mathbb{N}$ with $\frac{1}{N} < \tilde{\varepsilon}$.

Since the family $\tilde{F} = \{p_1/Q, \ldots, p_m/Q\}$ is strongly linearly independent over $\mathbb{Q}$, Step 3 above shows that Λ is an $(\tilde{F}, \mathbb{R}, \tilde{\varepsilon})$-matching set. We next see that Λ is also an $(F, \mathbb{R}, \varepsilon)$-matching set.

Let $a \in \mathbb{R}$. Since Λ is an $(\tilde{F}, \mathbb{R}, \tilde{\varepsilon})$-matching set, there is $x_a \in \Lambda_0$ such that $x_a \in N^{\lhd}(\tilde{F}, \tilde{\varepsilon}, a)$; that is, for each $i = 1, \ldots, m$, there are δ_i with $|\delta_i| < \tilde{\varepsilon}$ and $M_i \in \mathbb{Z}$ such that

$$\frac{p_i}{Q}(x_a) - \frac{p_i}{Q}(a) = \delta_i + M_i.$$

Then, for $j = m+1, \ldots, s$,

$$p_j(x_a) - p_j(a) = \sum_{i=1}^{m} Z_{ij} \left[\frac{p_i}{Q}(x_a) - \frac{p_i}{Q}(a)\right] = \sum_{i=1}^{m} Z_{ij}(\delta_i + M_i).$$

Since $\sum_{i=1}^{m} Z_{ij} M_i \in \mathbb{Z}$ and $\left|\sum_{i=1}^{m} Z_{ij}\delta_i\right| \le \varepsilon$, we see that $\psi_{p_j}(x_a) \in \psi_{p_j}(a) \cdot \mathcal{V}_\varepsilon$ for every $j = m+1, \ldots, s$. The same conclusion being obvious for $j = 1, \ldots, m$, it follows that $x_a \in N^{\lhd}(F, \varepsilon, a)$, as we wanted to show.

Having proved that Λ is an $(F, \mathbb{R}, \varepsilon)$-matching set for every $\varepsilon > 0$ and every finite set $F \subseteq \mathfrak{C}_n$, an application of Corollary 2.6 then concludes the proof. □

In the case of $\mathcal{AP}(\mathbb{R})$, we also obtain almost sure density.

Theorem 4.3 *Let $I^* = (I_k)_{k\in\mathbb{N}}$ be a sequence of intervals $I_k = [a_k, a_k + b_k] \subseteq \mathbb{R}$ with $\limsup(b_k)_{k\in\mathbb{N}} = +\infty$ and $t_k := \max(|a_k|, |a_k + b_k|) \ge k$ for every $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $\Lambda_k \subseteq I_k$ be a random subset with $|\Lambda_k| \ne O(\log t_k)$. Then, almost surely, $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is dense in $\mathbb{R}^{\mathcal{AP}}$.*

The same argument yields the following general version of Theorem 3.1 in [13].

Theorem 4.4 *Let $I^* = (I_k)_{k\in\mathbb{N}}$ be a sequence of intervals $I_k = [n_k, n_k + m_k] \subseteq \mathbb{Z}$ with $\limsup(m_k)_{k\in\mathbb{N}} = +\infty$ and $t_k := \max(|n_k|, |n_k + m_k|) \ge k$ for every $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $\Lambda_k \subseteq I_k$ be a random subset with $|\Lambda_k| \ne O(\log t_k)$. Then, almost surely, $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is dense in $\mathbb{Z}^{\mathcal{AP}}$.*

5 Bohr-Dense Sets with Special Properties

The estimates of Sect. 2 can be easily used to find Bohr-dense sets with special properties, as long as the conditions imposed in Lemma 2.12 and Theorem 4.2 leave enough room for a random subset to satisfy the required properties.

In this section we focus on interpolation properties of sets. For a given algebra $\mathcal{A} \subseteq C(G, \mathbb{C})$, with G a topological group, a subset $X \subseteq G$ is said to be an $\mathcal{A}$-*interpolation set* if every bounded function $f : X \to \mathbb{C}$ admits a continuous extension $\tilde{f} : G \to \mathbb{C}$ with $\tilde{f} \in \mathcal{A}$. We consider here a class of sets of interpolation for the algebra

$\mathcal{WAP}(G)$ of weakly almost periodic functions on G, which are those functions whose set of translates by elements of G is *weakly* relatively compact. One of the important features of this class is that the interpolation properties of its members are not a part of its definition, which focuses on its combinatorial side.

Definition 5.1 Let G be a topological group. A subset E of G is a t-set if for every $g \in G$, $g \neq 0$, the intersection $E \cap (E + g)$ is relatively compact.

The class of t-sets was introduced by Rudin in [17], where he proved that every function supported on a t-set of a discrete group is automatically weakly almost periodic, see [9] for further references on this topic. In the terminology of [9], t-sets in LCA groups are (approximable) $\mathcal{WAP}(G)$-interpolation sets.

Sets of interpolation for the Fourier–Stieltjes algebra $B(G)$ (consisting of Fourier–Stieltjes transforms of measures on $\widehat{G}$) are known as *Sidon sets* and have been heavily studied (see the monographs [10] and [15]). Since $B(G) \subseteq \mathcal{WAP}(G)$, we have that both Sidon sets and t-sets are sets of interpolation for the algebra of weakly almost periodic functions. An interesting observation is that every Sidon set can be decomposed as a finite union of t-sets (see [10, Corollary 6.4.7]).

As already noted in [13], the random process of Theorems 4.2 and 4.4 cannot be adapted to yield Sidon sets. It is well known, see, e.g., [10, Corollary 6.3.13], that a length N interval contains at most $C_E \log N$ elements of a Sidon set E. This route has however proved to be fruitful with other less demanding properties: Neuwirth [16], for instance, obtains dense subsets of $\mathbb{Z}^{\mathcal{AP}}$ that are $\Lambda(p)$ for every p, and Li, Queffélec and Rodríguez-Piazza [14] have obtained dense subsets of $\mathbb{Z}^{\mathcal{AP}}$ that are p-Sidon for every $p > 1$. We do not need these concepts here and refer to [10, 14, 16] for their proper definitions. It suffices to say that Sidon sets are p-Sidon for every $p > 1$ and $\Lambda(p)$-sets for every p.

While, as mentioned, our construction does not work with Sidon sets, it does work with the important class of t-sets. We show in this section that t-sets that are dense in $G^{\mathcal{AP}}$ do exist for $G = \mathbb{Z}$ and $G = \mathbb{R}$ and that, indeed, random constructions in the spirit of Lemma 2.12 lead almost surely to t-sets that are dense in $G^{\mathcal{AP}}$.

We first need a lemma that helps in recognizing t-sets. For a subset $\Lambda \subseteq \mathbb{R}$, we define its *step length* as

$$\mathrm{StL}(\Lambda) = \inf \left\{ |z - z'| \colon z, z' \in \Lambda, z \neq z' \right\}.$$

Lemma 5.2 *Let $([a_k, b_k])_{k\in\mathbb{N}}$ be a sequence of intervals in $\mathbb{R}$, and for every $k \in \mathbb{N}$, let Λ_k be a finite subset of $[a_k, b_k]$. If*

(1) *the sequence of gaps $(a_{k+1} - b_k)_{k\in\mathbb{N}}$ is increasing and unbounded, and*
(2) *there exists k_0 such that $\mathrm{StL}(\Lambda_k) > b_{k-1} - a_{k-1}$ for every $k \geq k_0$,*

then the set $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is a t-set.

Proof Suppose there is $0 \neq t_0 \in \mathbb{R}$ such that $\Lambda \cap (\Lambda + t_0)$ is unbounded. Since the sets Λ_k are finite, there will be $k < k'$, with $k > k_0$, $a_k - b_{k-1} > |t_0|$, and $t_k \in \Lambda_k \cap (\Lambda + t_0)$, $t_{k'} \in \Lambda_{k'} \cap (\Lambda + t_0)$. Then $t_0 = t_k - t_1 = t_{k'} - t_2$ with $t_1, t_2 \in \Lambda$. If $t_0 > 0$, using that $t_0 < a_k - b_{k-1} < a_{k'} - b_{k'-1}$, one has that $b_{k-1} < t_1 \leq t_k$ and that $b_{k'-1} < t_2 \leq t_{k'}$.

A symmetric argument works when $t_0 < 0$. It follows that $t_1 \in \Lambda_k$ and $t_2 \in \Lambda_{k'}$. But then $\mathrm{StL}(\Lambda_{k'}) \leq |t_{k'} - t_2| = |t_0| < |a_k - b_{k-1}|$, a contradiction with hypothesis (2). ☐

Theorem 5.3 *Let $I^* = (I_k)_{k\in\mathbb{N}}$ be a sequence of intervals $I_k = [z_k, z_k + N_k] \subseteq \mathbb{Z}$ such that the sequence of gaps $(z_{k+1} - z_k - N_k)_{k\in\mathbb{N}}$ is increasing and unbounded. For each $k \in \mathbb{N}$ let $\Lambda_k \subseteq I_k$ be a random subset with $|\Lambda_k| = \ell_k$. If $\sum_{k\in\mathbb{N}} \frac{\ell_k^2 N_{k-1}}{N_k} < \infty$, then, almost surely, $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is a t-set.*

Proof For $k \in \mathbb{N}$ we consider the event $\mathcal{B}_k = \{\Lambda_k \colon \mathrm{StL}(\Lambda_k) \leq N_{k-1}\}$. A rough estimate of the probability of this event is obtained by observing that any choice of ℓ_k elements with step length at most N_{k-1} is *witnessed* by elements in I_k at a distance of at most N_{k-1}. If for each element z in I_k we find those elements in I_k larger than z but within a distance of at most N_{k-1}, we see that there are at most N_{k-1} witnesses containing z. Since there are at most $\binom{N_k-1}{\ell_k-2}$ different subsets of I_k of cardinality ℓ_k that contain a given witness, we deduce altogether that

$$\mathbb{P}\left(\mathcal{B}_k\right) \leq \frac{(N_k+1)N_{k-1}\binom{N_k-1}{\ell_k-2}}{\binom{N_k+1}{\ell_k}} = \frac{(\ell_k-1)\ell_k N_{k-1}}{N_k} \leq \frac{\ell_k^2 N_{k-1}}{N_k}.$$

We conclude that $\sum_{k\in\mathbb{N}} \mathbb{P}\left(\mathcal{B}_k\right) < \infty$, and the Borel–Cantelli lemma then shows that, almost surely, there exists $k_0 \in \mathbb{N}$ such that $\Lambda_k \notin \mathcal{B}_k$ for every $k \geq k_0$. Lemma 5.2 then proves that Λ is a t-set. ☐

Theorem 5.4 *Let $I^* = (I_k)_{k\in\mathbb{N}}$ be a sequence of intervals $I_k = [z_k, z_k + N_k] \subseteq \mathbb{R}$ such that the sequence of gaps $(z_{k+1} - z_k - N_k)_{k\in\mathbb{N}}$ is increasing and unbounded. For each $k \in \mathbb{N}$ let $\Lambda_k \subseteq I_k$ be a random subset with $|\Lambda_k| = \ell_k$. If $\sum_{k\in\mathbb{N}} \frac{\ell_k^2 N_{k-1}}{N_k} < \infty$, then, almost surely, $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ is a t-set.*

Proof Again, for $k \in \mathbb{N}$ we consider the event $\mathcal{B}_k = \{\Lambda_k \colon \mathrm{StL}(\Lambda_k) \leq N_{k-1}\}$. We first estimate the probability of $\mathcal{B}_k$ when $\ell_k = 2$. For $\mathcal{B}_0 := \{\{x, y\} \subseteq I_k \colon |x - y| \leq N_{k-1}\}$ and considering the pair (x, y) to be uniformly distributed on the square $I_k \times I_k$, we obtain

$$\mathbb{P}(\mathcal{B}_0) = 1 - 2\int_{z_k+N_{k-1}}^{z_k+N_k}\int_{z_k}^{x-N_{k-1}} \frac{1}{N_k^2}\,dy\,dx = \frac{N_{k-1}(2N_k - N_{k-1})}{N_k^2}.$$

Now, if a set Λ_k consisting of ℓ_k points is chosen in I_k, for Λ_k to be in $\mathcal{B}_k$ it will be enough that any pair $\{x, y\}$ of its elements satisfies $|x - y| \leq N_{k-1}$. A rough estimate is then

$$\mathbb{P}\left(\mathcal{B}_k\right) \leq \binom{\ell_k}{2} \cdot \mathbb{P}\left(\mathcal{B}_0\right) = \frac{\ell_k(\ell_k-1)N_{k-1}}{2N_k^2}\left(2N_k - N_{k-1}\right) \leq \frac{\ell_k^2 N_{k-1}}{N_k}.$$

We deduce that $\sum_{k\in\mathbb{N}} \mathbb{P}\left(\mathcal{B}_k\right) < \infty$, and the Borel–Cantelli lemma then shows that, almost surely, there exists $k_0 \in \mathbb{N}$ such that $\Lambda_k \notin \mathcal{B}_k$ for every $k \geq k_0$. Lemma 5.2 proves then that Λ is a t-set. ☐

We now choose parameters in Theorem 4.2 so as to fit in Theorem 5.3.

Theorem 5.5 *Let $I^* = (I_k)_{k\in\mathbb{N}}$ be a sequence of intervals $I_k = [L_k, 2L_k] \subseteq \mathbb{R}$ with $L_k = (k!)^4$ for every $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $\Lambda_k \subseteq I_k$ be a random subset with $|\Lambda_k| = k$, and let $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$. Then, almost surely, Λ is a dense t-set in $\mathbb{R}^{\mathcal{AP}}$. If we choose $\Lambda_k \subseteq \mathbb{Z}$, then, almost surely, Λ is a dense t-set in $\mathbb{Z}^{\mathcal{AP}}$.*

Proof This sequence of intervals satisfies the hypotheses of both Theorems 4.2 and 5.4, or their analogs 4.4 and 5.3. Hence Λ almost surely satisfies both conclusions. □

The sets of Theorems 4.4 and 5.5 have asymptotic density zero. Other examples of Bohr-dense subsets of $\mathbb{Z}$ are obtained in [1]. The sets in [1] satisfy the condition

$$\lim_{N\to\infty} \frac{1}{N} |E_N \cap (E_N + k)| = 1, \text{ for all } k \in \mathbb{Z},$$

where E_N is the set consisting of the first N terms of E. They are therefore very far from being t-sets.

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